2/5/2019: Convergence, starting Newton's method. (Thurs)

Last time: we were concerned about existence of a fixed point $\xi = g(\xi)$. Actually, Brouwer's Fixed pt theorem gives us existence, so it's okay. But, there is another way to see existence.

Idea: Consider $x_{k+1} = g(x_k)$, with g a contraction.
Show the sequence $\{x_k\}$ is Cauchy. Cauchy $\Rightarrow$ Convergence.

What is "Cauchy"?

Definition: A sequence $\{x_k\}$ is Cauchy provided for each $\varepsilon > 0$, $\exists$ N so that for $m, n > N$ one has

$$|x_m - x_n| < \varepsilon.$$

Hint for homework:

To see $x_{k+1} = g(x_k)$ is Cauchy, for g a contraction, try to find a geometric series.

i.e. look at

$$|x_m - x_{m-1}| = |g(x_{m-1}) - g(x_{m-2})|$$

$$\leq L|x_{m-1} - x_{m-2}|$$

(By induction). $\leq L^{m-1}|x_1 - x_0|$.

Then, consider, for $m > n$,

$$|x_m - x_n|$$

$$= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq L^{m-1}|x_1 - x_0| + L^{m-2}|x_1 - x_0| + \dots + L^n|x_1 - x_0|$$

$$= |x_1 - x_0| \underbrace{\sum_{k=n}^{m-1} L^k}.$$

you can make this term really small...
why???

Another remark: why does Cauchy $\Rightarrow$ convergent?

i.e. If $\{x_k\}$ is Cauchy, then $x_k \to x^*$?

This is true by Bolzano-Weierstrass Thm: Each bounded sequence has a convergent subsequence.

Idea for proof:

1. Show that $\{x_k\}$ Cauchy implies $\{x_k\}$ is bounded.

2. By Bolzano-Weierstrass, extract a convergent subsequence $\{x_{n_k}\}$ so that $x_{n_k} \to x^*$.

3. Show that in fact $x_k \to x^*$.

By the way, this is true in $\mathbb{R}^n$,

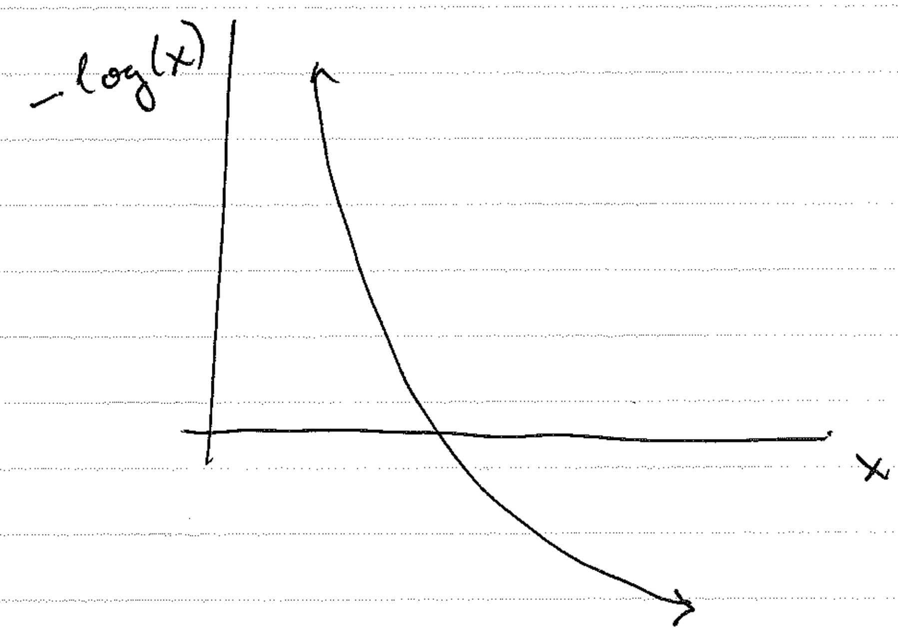
but this may not be true in a very general vector space.

In class before, we show that the smallest positive integer $k_0(\varepsilon)$ ensuring that $|x_k - \xi| \le \varepsilon$ must satisfy:

$$k_0(\varepsilon) \le \left\lfloor \frac{\log|x_0 - x_1| - \log((1-L)\varepsilon)}{\log(\frac{1}{L})} \right\rfloor + 1.$$

How does this upper bound change as we very L and ε?

As $L \to 1^-$, $-\log((1-L)\varepsilon) \to +\infty$



Also As $L \to 1^-$, $\log\left(\frac{1}{L}\right) \to 0^+$

So the upper bound to $k_0(L)$ gets really big for $L \approx 1$, or when g is really close to <u>NOT</u> being a contraction.

Also, as $\varepsilon \to 0$, $-\log((1-L)\varepsilon) \to +\infty$,

so smaller tolerances require more iterations...

<u>Example 1.3</u> We rewrote $f(x)=0$
$f(x) = e^x - 2x - 1$
as $x = \underbrace{\log(2x+1)}_{=g(x)}$, $x \in [1,2]$.

So we have $g'(x) = \dfrac{2}{2x+1}$, $x \in [1,2]$.

The largest g' gets on $[1,2]$ is $2/3$.

~~Thus~~ Choosing $L = 2/3$ can give us a worst case number of iterations to achieve a given tolerance.

For example, if $L = 2/3$, $\varepsilon = 0.5 \times 10^{-6}$

then $k_0(\varepsilon) \le \lfloor 32.78 \rfloor + 1 = 33$

<u>Example:</u> $f(x) = \sin x + e^{(x - 3\pi/2)}$

$$x \in \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right].$$

Clearly $f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) + 1$

$$= -1 + 1 = 0,$$

So our root is $\xi = \frac{3\pi}{2}$.

If we reformulate the root finding as a fixed pt problem, we get

$$0 = \sin x + e^{(x - 3\pi/2)}$$

$$\Leftrightarrow \quad -\sin x = e^{(x - 3\pi/2)}$$

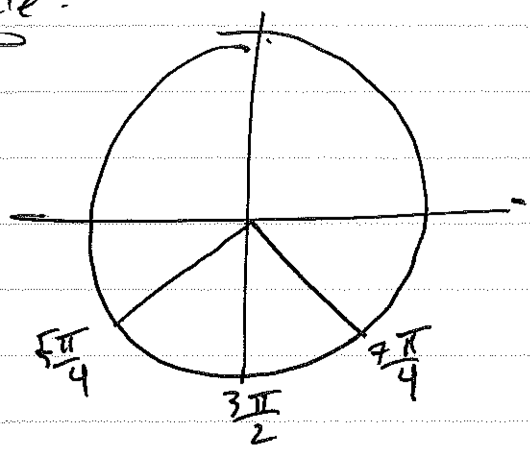
$$\Leftrightarrow \quad \log(-\sin x) = x - 3\pi/2$$

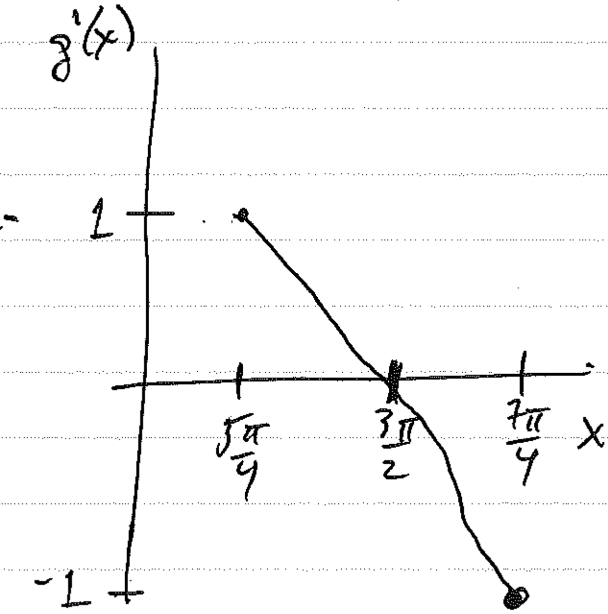
$$\Leftrightarrow \quad X - \underbrace{\log(-\sin x) - 3\pi/2}_{g(x).} = 0$$

Note that: $g'(x) = -\dfrac{\cos x}{\sin x}$,

$x \in \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$.

unit circle:





$|g'|$ is equal to 1 at the endpoints of the interval, but it is a contraction locally around $\xi = \frac{3\pi}{2}$.

A motivation for a local version of the contraction mapping theorem.

Theorem 1.5 $g : [a,b] \to [a,b]$, continuous let $\xi = g(\xi) \in [a,b]$.

Assume g has a continuous derivative in some neighborhood of ξ with $|g'(\xi)| < 1$, Then the iteration $\longrightarrow$

$$x_{k+1} = g(x_k)$$

converges to ξ, provided the initial guess x_0 is sufficiently close to ξ.

Pf: The crucial assumptions are that $|g'(\xi)| < 1$ and the derivative is cnts in a neighborhood
(continuous)
of ξ.

Neighborhood just means an interval around ξ, i.e

$$[\xi - h, \xi + h]$$

For some $h > 0$.

Continuity at ξ means that for each $\varepsilon > 0$, $\exists\ \delta > 0$ so that

$$|x - \xi| < \delta \Rightarrow |g'(x) - g'(\xi)| < \varepsilon.$$

Apply continuity with $\varepsilon = \frac{1}{2}(1 - |g'(\xi)|)$.

So $\exists\, \delta > 0$, and $\delta \le h$, so that

$$|x - \xi| < \delta \Rightarrow |g'(x) - g'(\xi)| < \frac{1}{2}(1 - |g'(\xi)|).$$

using the Δ - inequality:

$$\begin{aligned} |g'(x)| &= |g'(x) - g'(\xi) + g'(\xi)| \\ &\le |g'(x) - g'(\xi)| + |g'(\xi)| \\ &< \frac{1}{2}(1 - |g'(\xi)|) + |g'(\xi)|. \\ &= \frac{1}{2}(1 + |g'(\xi)|) \end{aligned}$$

Define $L = \frac{1}{2}(1 + |g'(\xi)|)$.

Note that $L < 1$.

We have shown that for

$x \in [-\delta + \xi, \xi + \delta]$, $|g'(x)| \le L < 1$.

So g is "locally" a contraction.

Suppose $x_k \in (\xi - \delta, \xi + \delta)$.

By definition:

$$x_{k+1} - \xi = g(x_k) - \xi = g(x_k) - g(\xi)$$
$$= (x_k - \xi)\, g'(\eta)$$

By the Mean Value Theorem,

where ~~η ...~~ η is in between x_k and ξ.

Note that $|x_{k+1} - \xi| = |(x_k - \xi)\, g'(\eta)|$

$$\leq L\, |x_k - \xi|$$

$$< |x_k - \xi|,$$

So if $x_k \in (\xi - \delta, \xi + \delta)$, then

x_{k+1} is also. By induction, this shows the sequence $\{x_k\}$ is in the interval $[\xi - \delta, \xi + \delta]$ if x_0 is also. The sequence will converge since g is a contraction on $[\xi - \delta, \xi + \delta]$.

Definition 1.3

$g : [a, b] \to [a, b]$, continuous.

$\xi = g(\xi) \in [a, b]$.

$x_{k+1} = g(x_k)$.

ξ is called a <u>stable fixed point</u> of g if $x_k \to \xi$ whenever x_0 is sufficiently close to ξ.

ξ is called an <u>unstable fixed point</u> of g if x_k does not converge for any starting value x_0 sufficiently close to ξ (except for $x_0 = \xi$).

<u>Remark</u>: By local version of contraction mapping, if $\xi = g(\xi)$, g' is continuous in a neighborhood of ξ, and $|g'(\xi)| < 1$, then ξ is a stable fixed pt.

Remark: Look at ratios of "error" between ~~x_k~~ x_k and ξ.

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \lim_{k\to\infty} \left| \frac{g(x_k) - g(\xi)}{x_k - \xi} \right|$$

$$= |g'(\xi)|,$$

so $|g'|$ can tell us something about how quickly these iterates converge.

Definition 1.4

Suppose $\xi = \lim_{k\to\infty} x_k$.

$x_k \to \xi$ at least linearly provided

there exists a sequence $\{\varepsilon_k\}$ of positive real #'s, converging to 0, and $\mu \in (0,1)$, so that

$$|x_k - \xi| \le \varepsilon_k \quad \text{and}$$

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu.$$

If $\mu = 0$, x_k is said to converge <u>super linearly</u>.

If $\mu \in (0,1)$ and $\varepsilon_k = |x_k - \xi|$, then x_k is said to converge <u>linearly</u>.

Also $\rho = -\log \mu$ is called the <u>asymptotic rate of convergence</u>.

If $\mu = 1$ and $\varepsilon_k = |x_k - \xi|$, x_k is said to converge <u>sublinearly</u>.

<u>Example</u>: $x_k = \frac{1}{2^k}$, $\xi = 0$.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{1}{2}$, so x_k converges linearly.

Example: $x_k = \frac{1}{10^k}$, $\xi = 0$.

$$1,\ 0.1,\ 0.01,\ 0.001,\ \ldots$$

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{1}{10},$$ linear convergence.

look at $\rho = -\log_{10}\mu = -\log_{10}\left(\frac{1}{10}\right)$

$$= \log_{10}(10) = 1.$$

Every iteration we gain 1 more digit of accuracy.

If $\rho \approx \dfrac{\text{\# of digits of accuracy}}{1 \text{ iteration}}$

then $\left\lfloor \dfrac{1}{\rho} \right\rfloor + 1 \approx \dfrac{\text{\# of iterations}}{1 \text{ digit of accuracy}}$