1/29/2019 : Solution of equations by iteration.

We know how to solve

$$a x = b , \quad a \neq 0.$$

as

$$x = \frac{b}{a}.$$

Also, we know how to solve

$$a x^2 + b x + c = 0 \quad a \neq 0.$$

$$x^{\pm} = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}.$$

What if we don't know, or have, a closed form solution... more generally to

$$f(x) = 0 ?$$

Framework: $f : [a, b] \to \mathbb{R}$,

and we are interested in solutions ξ such that $\xi \in \mathbb{R}$ and

$$f(\xi) = 0.$$

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<u>Theorem</u> (1.1) Let f be a real valued Function with $f : [a,b] \to \mathbb{R}$ and continuous. If $f(a)f(b) \leq 0$, then there exists $\xi \in [a,b]$ such that $f(\xi)=0$.

<u>Proof</u>: If $f(a) = 0$ or $f(b) = 0$, we are done, since $\xi = a$ or $\xi = b$.

Assume then that both $f(a) \neq 0$ and $f(b) \neq 0$, so $f(a)f(b) \neq 0$. Since we are assuming that $f(a)f(b) \leq 0$ we must have

$$f(a)\, f(b) < 0.$$

This implies that $f(a)$ and $f(b)$ have opposite sign. Without loss of generality, we have

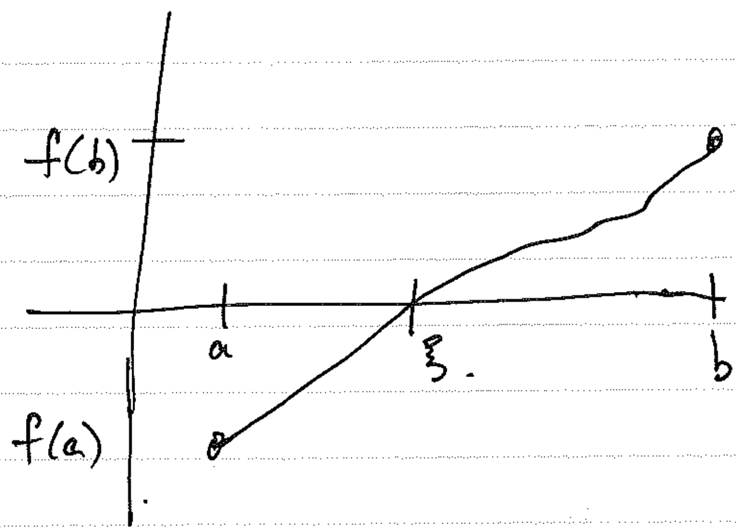
$$f(a) < 0 < f(b),$$

and by the intermediate value theorem, $\exists\ \xi \in [a,b]$ so that

$$f(\xi) = 0.$$

This completes the proof.

Example: (Sketch)



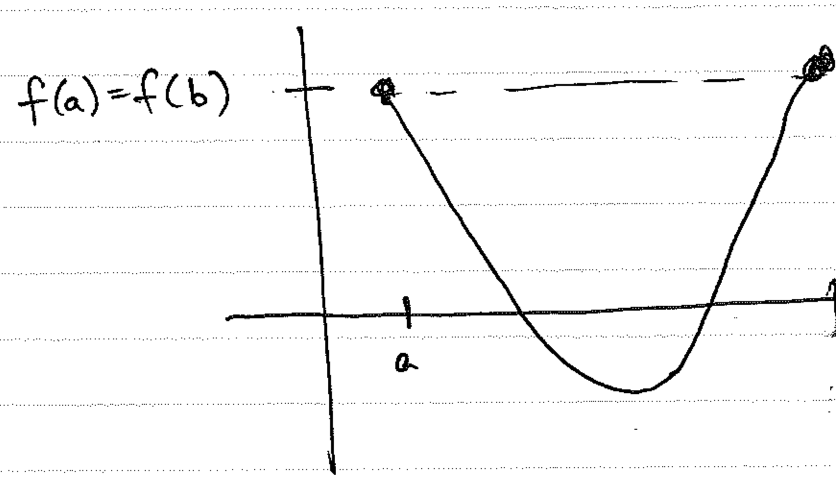
$$\begin{cases} f(a) < 0 \\ f(b) > 0 \\ f(\xi) = 0. \end{cases}$$

What about the converse of Thm 1.1?

Converse of Thm 1.1

If there exists $\xi \in [a,b]$ such that $f(\xi) = 0$, then $f(a)\,f(b) \le 0$.

This is not true! Counterexample: (?)



$f(a)f(b) > 0.$

If you can rewrite $f(x) = 0$ as $x - g(x) = 0$, for some function g, you can think of ξ such that $f(\xi) = 0$ as a <u>fixed point</u> of g.

$$f(\xi) = 0 \iff \xi = g(\xi).$$

(provided $f(x) = 0 \iff x - g(x) = 0$)

A <u>very important</u> theorem:

<u>Thm</u> (1.2) Brouwer's Fixed Point Thm.

Suppose g is a real-valued, continuous function

$$g : [a,b] \to [a,b].$$

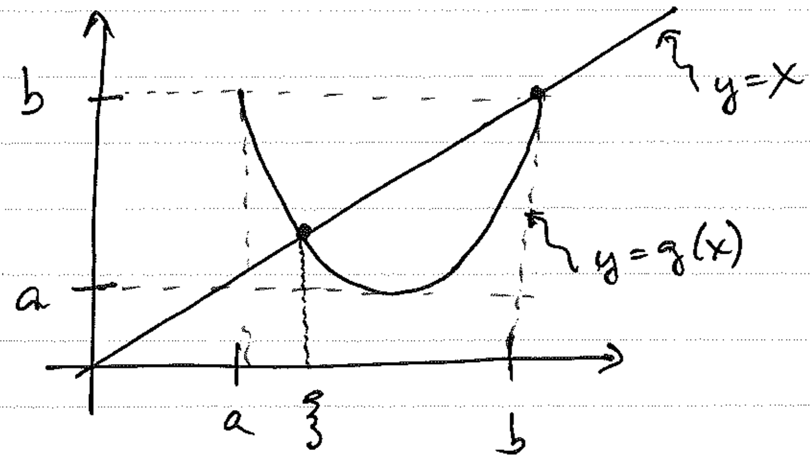
i.e. $g(x) \in [a,b]$ for all $x \in [a,b]$

Then there exists $\xi \in [a,b]$ such that $\xi = g(\xi)$. ξ is called a <u>fixed point</u> of g.

<u>Pf</u>: Define $f(x) = x - g(x)$.

Note that $f(a) = a - g(a) \leq 0$ and $f(b) = b - g(b) \geq 0$ since $g(a), g(b) \in [a,b]$.

Thus $f(a)f(b) \leq 0$. By Thm 1.1, $\exists$ $\xi \in [a,b]$ so that $0 = f(\xi) = \xi - g(\xi)$.



Example: $\begin{cases} f(x) = e^x - 2x - 1. \\ f : [1,2] \to \mathbb{R} \end{cases}$

First, check conditions of Thm 1.1.

See that $f(1) = e - 2 - 1 = e - 3 < 0$ and
$f(2) = e^2 - 3 > 0.$
$(e \approx 2.718)$

So by Thm 1.1, $\exists\ \xi \in [1,2]$ so that

$$f(\xi) = 0.$$

We can also try to use Thm 1.2 to see this. We need to write $f(x) = 0$ in a form $x - g(x) = 0$. An easy way is to:

$$e^x - 2x - 1 = 0 \iff x - \underbrace{\left(\frac{e^x - 1}{2}\right)}_{= g(x)} = 0.$$

with $g(x) = \frac{e^x - 1}{2}$, note that

$g(1) = \frac{e-1}{2} < 1$, since $e \approx 2.718\ldots$

So in particular $g(x) \notin [1,2] \ \forall x \in [1,2]$.

So, we <u>cannot</u> apply Thm 1.2 with this choice of g.

Instead, take a logarithm.

$$f(x) = e^x - 2x - 1 = 0$$

$$\Leftrightarrow \quad e^x = 2x + 1$$

$$\Leftrightarrow \quad x = \log(2x+1).$$

$$\Leftrightarrow \quad x - \underbrace{\log(2x+1)}_{= g(x).} = 0$$

In this case, $g(1) = \log(3) \approx 1.099$

$g(2) = \log(5) \approx 1.609\ldots$

And since $g(x)$ is monotone

increasing, $g(x) \in [1,2]$ $\forall x \in [1,2]$.

So, we can apply Thm 1.2 to conclude that $\exists$ $\xi \in [1,2]$ so that

$$\xi = g(\xi) \iff f(\xi) = 0.$$

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<u>Definition</u> (1.1) Suppose g is a real-valued function $g: [a,b] \to [a,b]$.

Given some initial guess $x_0 \in [a,b]$,

$$x_{k+1} = g(x_k), \quad k = 0,1,2,\dots$$

is called a <u>simple iteration</u> or <u>fixed-point iteration</u>. the numbers x_k are the iterates.

<u>Exercise</u>: Suppose the simple iteration sequence converges. Show that its limit is a fixed point of g.
<u>Hint</u>: use continuity of g.

We want some criteria to ensure the sequence $x_{k+1} = g(x_k)$ converges.

This criteria will necessarily be a condition on the function g.

<u>Definition</u>: g is a contraction on $[a,b]$ provided there exists a constant $0 < L < 1$ so that

$$|g(x) - g(y)| \le L|x-y| \quad \forall x,y \in [a,b].$$

Remark: When L is any positive constant, not necessarily less than 1, the condition $|g(x)-g(y)| \le L|x-y|$ is called Lipschitz continuity.

Thm: (1.3) (Contraction mapping)

Also Very Important!

Suppose $g : [a,b] \to [a,b]$ is a contraction. Then g has a unique fixed point $\xi \in [a,b]$, and the sequence

$$x_{k+1} = g(x_k)$$

converges to ξ for any $x_0 \in [a,b]$.

pf:

Why is ξ unique? Assume for the sake of contradiction that $\exists\ \eta,\ \eta \neq \xi$, so that $g(\eta)=\eta$.

Then $|g(\eta)-g(\xi)| \le L|\eta-\xi|$

since g is a contraction. But, since $L<1$, we have:

$$|\eta - \xi| = |g(\eta) - g(\xi)| \le L|\eta - \xi|$$

$$\rightsquigarrow (1-L)|\eta - \xi| \le 0$$

which is a contradiction since

$$|\eta - \xi| > 0 \text{ and } (1-L) < 0.$$

Now, we show that $x_k \to \xi$.

We have, by definition:

$$|x_k - \xi| = |g(x_{k-1}) - g(\xi)|$$

$$\le L|x_{k-1} - \xi|, \quad k \geq 1.$$

By induction, we have

$$|x_k - \xi| \le L^k |x_0 - \xi|, \quad k \geq 1.$$

So $\lim_{k\to\infty} |x_k - \xi| \le \lim_{k \to \infty} L^k |x_0 - \xi| = 0.$

Showing $\lim_{k\to\infty} x_k = \xi$.

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