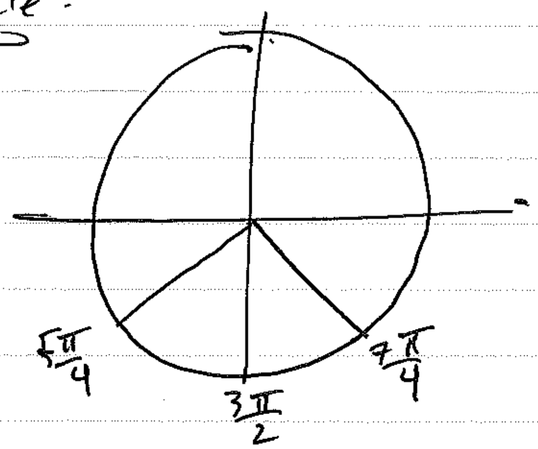
2/7/2019:

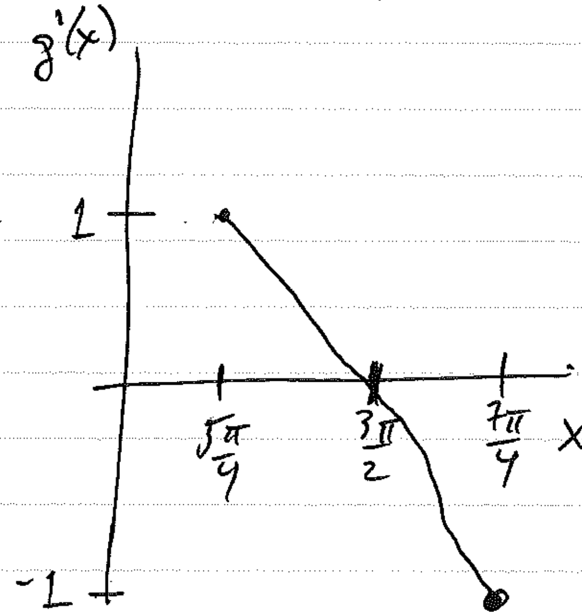
Stability, local contraction mapping, and Newton's method

Note that: $g'(x) = -\dfrac{\cos x}{\sin x}$,

$x \in \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$.

unit circle:





$|g'|$ is equal to 1 at the endpoints of the interval, but it is a contraction locally around $\xi = \frac{3\pi}{2}$.

A motivation for a local version of the contraction mapping theorem.

starting from here.

Theorem 1.5 $g : [a, b] \to [a, b]$, continuous, let $\xi = g(\xi) \in [a, b]$.

Assume g has a continuous derivative in some neighborhood of ξ with $|g'(\xi)| < 1$, Then the iteration $\longrightarrow$

$$x_{k+1} = g(x_k)$$

converges to ξ, provided the initial guess x_0 is sufficiently close to ξ.

<u>Pf</u>: The crucial assumptions are that $|g'(\xi)| < 1$ <u>and</u> the derivative is cnts (continuous) in a neighborhood of ξ.

Neighborhood just means an interval around ξ, i.e

$$[\xi - h, \xi + h]$$

For some $h > 0$.

<u>Continuity</u> at ξ means that for each $\varepsilon > 0$, $\exists\, \delta > 0$ so that

$$|x - \xi| < \delta \Rightarrow |g'(x) - g'(\xi)| < \varepsilon.$$

Apply continuity with $\varepsilon = \frac{1}{2}\left(1 - |g'(\xi)|\right)$.

So $\exists\, \delta > 0$, and $\delta \le h$, so that

$$|x - \xi| < \delta \Rightarrow |g'(x) - g'(\xi)| < \frac{1}{2}\left(1 - |g'(\xi)|\right).$$

Using the Δ - inequality:

$$\begin{aligned} |g'(x)| &= |g'(x) - g'(\xi) + g'(\xi)| \\ &\le |g'(x) - g'(\xi)| + |g'(\xi)| \\ &< \frac{1}{2}\left(1 - |g'(\xi)|\right) + |g'(\xi)|. \\ &= \frac{1}{2}\left(1 + |g'(\xi)|\right) \end{aligned}$$

Define $L = \frac{1}{2}\left(1 + |g'(\xi)|\right)$.

Note that $L < 1$.

We have shown that for

$x \in [-\delta + \xi, \xi + \delta]$, $|g'(x)| \le L < 1$.

So g is "locally" a contraction.

Suppose $x_k \in (\xi - \delta, \xi + \delta)$.

By definition:

$$x_{k+1} - \xi = g(x_k) - \xi = g(x_k) - g(\xi)$$
$$= (x_k - \xi)\, g'(\eta)$$

By the Mean Value Theorem, where ~~...~~ η is in between x_k and ξ.

Note that $|x_{k+1} - \xi| = |(x_k - \xi)\, g'(\eta)|$

$$\leq L\, |x_k - \xi|$$

$$< |x_k - \xi|,$$

So if $x_k \in (\xi - \delta, \xi + \delta)$, then x_{k+1} is also. By induction, this shows the sequence $\{x_k\}$ is in the interval $[\xi - \delta, \xi + \delta]$ if x_0 is also. The sequence will converge since g is a contraction on $[\xi - \delta, \xi + \delta]$.

Definition 1.3

$g : [a, b] \to [a, b]$, continuous.

$\xi = g(\xi) \in [a, b]$.

$x_{k+1} = g(x_k)$.

ξ is called a <u>stable fixed point</u> of g if $x_k \to \xi$ whenever x_0 is sufficiently close to ξ.

ξ is called an <u>unstable fixed point</u> of g if x_k does not converge for any starting value x_0 sufficiently close to ξ (except for $x_0 = \xi$).

Remark: By local version of contraction mapping, if $\xi = g(\xi)$, g' is continuous in a neighborhood of ξ, and $|g'(\xi)| < 1$, then ξ is a stable fixed pt.

Remark: Look at ratios of "error" between ~~x_k~~ x_k and ξ.

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \lim_{k\to\infty} \left| \frac{g(x_k) - g(\xi)}{x_k - \xi} \right|$$

$$= |g'(\xi)|,$$

so $|g'|$ can tell us something about how quickly these iterates converge.

Definition 1.4

Suppose $\xi = \lim_{k\to\infty} x_k$.

$x_k \to \xi$ at least linearly provided

there exists a sequence $\{\varepsilon_k\}$ of positive real #'s, converging to 0, and $\mu \in (0,1)$, so that

$$|x_k - \xi| \le \varepsilon_k \text{ and}$$

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu.$$

If $\mu = 0$, x_k is said to converge <u>superlinearly</u>.

If $\mu \in (0,1)$ and $\varepsilon_k = |x_k - \xi|$, then x_k is said to converge <u>linearly</u>.

Also $\rho = -\log \mu$ is called the <u>asymptotic rate of convergence</u>.

If $\mu = 1$ and $\varepsilon_k = |x_k - \xi|$, x_k is said to converge <u>sublinearly</u>.

<u>Example</u>: $x_k = \dfrac{1}{2^k}$, $\xi = 0$.

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$$

$\lim\limits_{k \to \infty} \dfrac{\varepsilon_{k+1}}{\varepsilon_k} = \dfrac{1}{2}$, so x_k converges linearly.

Example: $x_k = \frac{1}{10^k}$, $\xi = 0$.

$1, 0.1, 0.01, 0.001, \ldots$

$\lim_{k \to \infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{1}{10}$, linear convergence.

look at $\rho = -\log_{10} \mu = -\log_{10}\left(\frac{1}{10}\right)$

$= \log_{10}(10) = 1.$

Every iteration we gain 1 more digit of accuracy.

If $\rho \approx \dfrac{\#\text{ of digits of accuracy}}{1 \text{ iteration}}$

then $\left\lfloor \frac{1}{\rho} \right\rfloor + 1 \approx \dfrac{\#\text{ of iterations}}{1 \text{ digit of accuracy}}$

Theorem 1.6 Suppose $\xi = g(\xi)$,

g has a continuous derivative in some neighborhood of ξ, and let $|g'(\xi)| > 1$.

Then the sequence $\{x_k\}$ defined by $x_{k+1} = g(x_k)$ does not converge to ξ. for ANY starting value x_0, $x_0 \neq \xi$.

Pf: Suppose $x_0 \neq \xi$. Determine an interval $[\xi - \delta, \xi + \delta]$, $\delta > 0$ where $|g'(x)| \geq L > 1$, For some CONSTANT L.

If $x_k \in [\xi - \delta, \xi + \delta]$, we have

$$|x_{k+1} - \xi| = |g(x_k) - g(\xi)|$$

$$= |(x_k - \xi)\, g'(\eta_k)|$$

$$\geq L\, |x_k - \xi|.$$

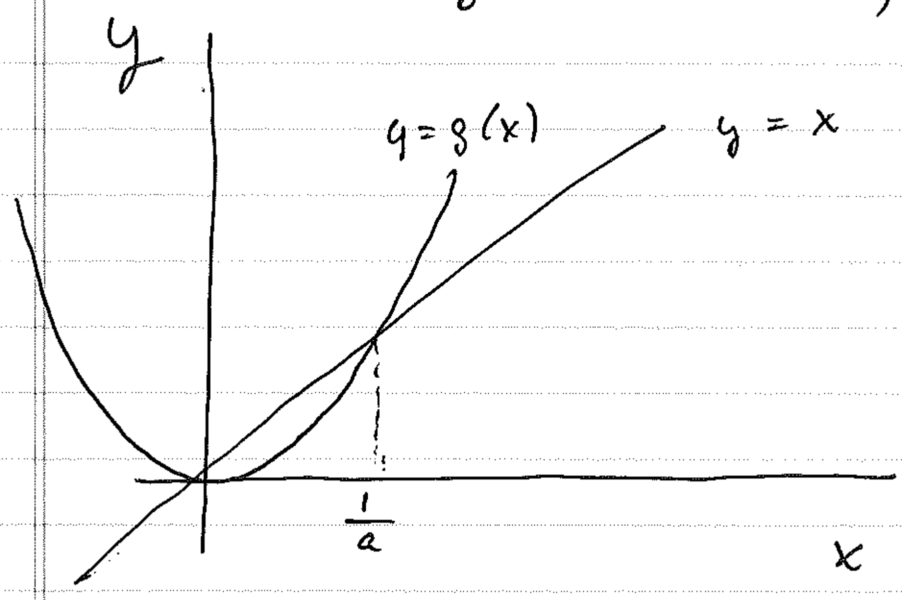
For η_k between x_k and ξ.

Since $L > 1$, after a finite number of steps, the sequence leaves the interval $[\xi - \delta, \xi + \delta]$. So it cannot converge.

Example: Stable and unstable Fixed points.

Consider fixed pts of the function

$$g(x) = ax^2, \quad a \neq 0.$$



Pictorially, we can see g has two fixed points, $\xi_1 = 0$ and

$$\xi_2 = \frac{1}{a}.$$

Note that around 0, $|g'(x)| < 1$. To see that, differentiate g:

$$g'(x) = 2ax.$$

So $|g'(x)| = 2a|x|$.

$\longrightarrow$

If $|x| < \frac{1}{2a}$, then

$$|g'(x)| = 2a|x| < 1.$$

IN particular this shows that $\xi_1 = 0$ is a stable fixed point for g.

What about $\xi_2 = \frac{1}{a}$?

$$g'(\xi_2) = 2ax\Big|_{x=\xi_2} = 2,$$

so ξ_2 is an unstable fixed point, i.e. the simple iteration

$$x_{k+1} = g(x_k)$$

will not converge to ξ_2 !!!

Definition 1.5 (Relaxation)

Suppose f is a real valued function, defined and continuous in a neighborhood of a real number ξ. "Relaxation" is the following iteration:

$$x_{k+1} = x_k - \lambda f(x_k), \quad k \geq 0.$$

where $\lambda \neq 0$ is a fixed real number, and x_0 is some initial value which will probably have to be close to ξ.

Remark: Notice that if $x_k \longrightarrow \xi$,

Then the relaxation iteration converges to a root of f,

$$\begin{array}{ccccc} x_{k+1} & = & x_k & - \lambda & f(x_k) \\ \downarrow & & \downarrow & & \downarrow \\ \xi & & \xi & & \xi \end{array}$$

$$\Rightarrow \quad \underline{\lambda f(\xi) = 0}.$$

Thm: 1.7.

f real valued, defined and continuous in a neighborhood of ξ.

$f(\xi) = 0$.

f' is also defined and continuous in a neighborhood of ξ, and

$f'(\xi) \neq 0$.

Then $\exists\ \lambda, \delta > 0$ so that the sequence $\{x_k\}$ defined by

$$x_{k+1} = x_k - \lambda f(x_k)$$

converges to ξ for ANY x_0 in the interval $[\xi - \delta, \xi + \delta]$.

Pf: WLOG, suppose $f'(\xi) = \alpha > 0$.

By continuity of f', $\exists\ \delta > 0$ so that

$$f'(x) \geq \frac{1}{2}\alpha \quad \text{for } x \in [\xi - \delta, \xi + \delta].$$

Take M to be an upper bound for f' in $[\xi-\delta, \xi+\delta]$.

So we have $\frac{1}{2}\alpha \le f'(x) \le M$, $x \in [\xi-\delta, \xi+\delta]$.

$$\Leftrightarrow \quad 1-\lambda M \le 1-\lambda f'(x) \le 1-\frac{1}{2}\lambda\alpha$$
For $x \in [\xi-\delta, \xi+\delta]$.

<u>Idea</u>: Pick ν satisfying

$$\begin{array}{l} (1) \\ (2) \end{array} \left\{ \begin{array}{l} 1-\lambda M = -\nu \\ 1-\frac{1}{2}\lambda\alpha = \nu \end{array} \right\}$$

(1) $\Leftrightarrow$ $\lambda M - 1 = \nu$. Plug this into (2)

$$\Rightarrow \quad 1-\frac{1}{2}\lambda\alpha = \lambda M - 1$$

Solving for λ, we get:

$$2-\lambda\alpha = 2\lambda M - 2$$

$$\Leftrightarrow \quad 4 = \lambda(\alpha+2M)$$

$$\Leftrightarrow \quad \lambda = \frac{4}{\alpha+2M}$$

and

$$\nu = \lambda M - 1$$

$$= \frac{4M}{\alpha + 2M} - 1$$

$$= \frac{4M}{\alpha + 2M} - \frac{\alpha + 2M}{\alpha + 2M}$$

$$\boxed{\nu = \frac{2M - \alpha}{2M + \alpha}.}$$

In summary, we have chosen λ so that $|1 - \lambda f'(x)| \le \nu < 1$.
For $x \in [\xi - \delta, \xi + \delta]$.
But if we define $g(x) = x - \lambda f(x)$,
then we have $|g'(x)| < 1$ for $x \in [\xi - \delta, \xi + \delta]$.
So the iteration $x_{k+1} = x_k - \lambda f(x_k)$
converges for $x_0 \in [\xi - \delta, \xi + \delta]$, for
the above choice of λ.

What if we allowed the relaxation parameter to depend on x_k?

$$x_{k+1} = x_k - \lambda(x_k) f(x_k).$$

What should $\lambda(x_k)$ be?

By the previous theorem, convergence of the relaxation <u>depends on $g'(\xi)$</u>.

If $g(x) = x - \lambda(x) f(x)$

Then $g'(x) = 1 - \lambda'(x) f(x) - \lambda(x) f'(x)$

Plugging in ξ, and using $f(\xi) = 0$, we have:

$$g'(\xi) = 1 - \lambda'(\xi) f(\xi) - \lambda(\xi) f'(\xi)$$

$$= 1 - \lambda(\xi) f'(\xi).$$

If $\lambda(\xi) = \dfrac{1}{f'(\xi)}$, then $g'(\xi) = 0$,

i.e. g' is very small!!!

This suggests us to try

$$\lambda(x_k) = \frac{1}{f'(x_k)}.$$

<u>Definition 1.6</u> (Important!)

This is <u>Newton's method</u>:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

We assume that $f'(x_k) \neq 0$ for all $k \geq 0$.

<u>Example</u>: consider again the function $g(x) = ax^2$, $a \neq 0$.
Before, we were interested in fixed points of g, but had trouble dealing with the fixed point $\xi_2 = \frac{1}{a}$.

Let's try to apply Newton's method to

$$f(x) = x - g(x)$$

$$f'(x) = 1 - 2ax$$

$\longrightarrow$

So, $f'(\xi_2) = 1 - 2a\,\xi_2$

$$= 1 - 2a\left(\frac{1}{a}\right) = -1.$$

So we are "in the clear" to apply Newton's method, since $f'(\xi_2) \neq 0$.

The iteration looks like.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{a x_k^2}{(1 - 2 a x_k)}.$$

Definition: The secant method is defined by the following iteration

$$x_{k+1} = x_k - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right).$$

$$k \geq 1.$$

x_0 and x_1 are starting values.